

COUNTING FIXED POINTS, TWO-CYCLES, AND COLLISIONS OF THE DISCRETE EXPONENTIAL FUNCTION USING p -ADIC METHODS

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ABSTRACT. Brizolis asked for which primes p greater than 3 does there exist a pair (g, h) such that h is a fixed point of the discrete exponential map with base g , or equivalently h is a fixed point of the discrete logarithm with base g . Zhang (1995) and Cobeli and Zaharescu (1999) answered with a “yes” for sufficiently large primes and gave estimates for the number of such pairs when g and h are primitive roots modulo p . In 2000, Campbell showed that the answer to Brizolis was “yes” for all primes. The first author has extended this question to questions about counting fixed points, two-cycles, and collisions of the discrete exponential map. In this paper, we use p -adic methods, primarily Hensel’s lemma and p -adic interpolation, to count fixed points, two cycles, collisions, and solutions to related equations modulo powers of a prime p .

1. INTRODUCTION

The idea of counting fixed points of discrete exponential functions is usually traced back to Demetrios Brizolis (see [12, Paragraph F9]), who asked whether, given a prime $p > 3$, there is always a pair (g, x) such that g is a primitive root modulo p , $g, x \in \{1, \dots, p-1\}$, and

$$(1) \quad g^x \equiv x \pmod{p} ?$$

We can regard solutions to this equation as fixed points of a discrete exponential function. Wen-Peng Zhang ([20]) proved that the answer to Brizolis’ question was always yes for sufficiently large p , a result which was rediscovered independently by Cobeli and Zaharescu in [6]. Mariana (Campbell) Levin proved the result for all primes in [5]. (See also [19].)

Zhang (and independently Cobeli and Zaharescu) also provided a way of estimating the number of pairs (g, x) which satisfy the conditions above and also have x being a primitive root. Specifically, if $N(p)$ is the number of such pairs given a prime p , we have:

Theorem 1 (Zhang, independently by Cobeli and Zaharescu).

$$\left| N(p) - \frac{\phi(p-1)^2}{p-1} \right| \leq d(p-1)^2 \sqrt{p}(1 + \ln p),$$

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where $d(p-1)$ is the number of divisors of $p-1$.

The first author, in [13, 14] investigated the problem of counting the number of solutions to Brizolis' conditions when g and x are not necessarily primitive roots. If $F(p)$ is the number of such pairs (g, x) , it was conjectured that

$$F(p) \sim (p-1)$$

as p goes to infinity. It was proved by the first author and Pieter Moree in [15, Thm. 4.9] that this is true for a set of primes of positive relative density. Bourgain, Konyagin, and Shparlinski proved in [3] that the conjecture is true for a set of primes of relative density 1. The same authors proved in [4], that a weaker result, $F(p) = O(p)$, is true for all p , and also that $F(p) \geq (p-1) - o(p)$ for all p .

This paper was motivated by the attempt to similarly count solutions (g, x) to the equation

$$(2) \quad g^x \equiv x \pmod{p^e}$$

with $g, x \in \{1, \dots, p^e\}$, $p \nmid g$ and $p \nmid x$. Based on numerical evidence, we conjecture that the number of these solutions is asymptotically equivalent to $p^{e-1}(p-1)$ as p goes to infinity, and furthermore that the number of solutions with $g \equiv i$ modulo p is asymptotically equivalent to p^{e-1} for any i as p goes to infinity. We would expect that the techniques used to prove the theorems above could also be applied to this case.

We then attempted to investigate the situation as p is held fixed and e goes to infinity. This led naturally to an examination of the function $x \mapsto g^x$ where g is fixed and x ranges through the p -adic integers \mathbb{Z}_p , which is carried out in Sections 2 and 3. The (perhaps) surprising discovery is what happens when we look for solutions x to (2) not in the set $\{1, \dots, p^e\}$ but rather in the “correct” set $\{1, \dots, p^e m\}$, where m is the multiplicative order of g modulo p . We show in Section 4 that the number of solutions in this more natural setting is exactly what one would expect from our conjectures, with no error term. (In the case $e = 1$, [19] observes that it is easy to find fixed points outside the set $\{1, \dots, p\}$ but does not explicitly count them.) Lev Glebsky, in [9], proves a similar result to ours in the case where $m = p-1$ using a very different method.¹

The papers [13–15] also investigated three related questions: the number of two-cycles of the discrete exponential function, or solutions to

$$(3) \quad g^h \equiv a \pmod{p} \quad \text{and} \quad g^a \equiv h \pmod{p},$$

the number of solutions to a discrete self-power equation

$$(4) \quad x^x \equiv c \pmod{p}$$

for fixed c , and the number of collisions of the discrete self-power function, i.e., solutions to

$$(5) \quad h^h \equiv a^a \pmod{p}.$$

It was conjectured in these papers that the number of solutions $T(p)$ to (3) with $1 \leq g, h, a \leq p-1$ and $h \not\equiv a$ modulo p was

$$T(p) \sim (p-1),$$

¹Our thanks to Igor Shparlinski for this reference.

the number of solutions $S(p; c)$ to (4) with $1 \leq x \leq p-1$ was

$$S(p; c) \sim \sum_{d \mid \frac{p-1}{m}} \frac{\phi(dm)}{dm}$$

where m is the order of c modulo p , and the number of solutions $C(p)$ to (5) with $1 \leq h, a \leq p-1$ and $h \not\equiv a$ modulo p was

$$C(p) \sim \sum_{m \mid p-1} \phi(m) \left(\sum_{d \mid \frac{p-1}{m}} \frac{\phi(dm)}{dm} \right)^2 = \sum_{d \mid p-1} \frac{J_2(d)}{d},$$

where $J_2(n) = n^2 \prod_{p \mid n} (1 - p^{-2})$ is Jordan's totient function, which counts the number of pairs of positive integers all less than or equal to n that form a mutually coprime triple together with n . Balog, Broughan, and Shparlinski ([1]) showed the weaker statements that $S(p; c) \leq p^{1/3+o(1)} m^{2/3}$ and $S(p; c) \leq p^{1+o(1)} m^{-1/12}$, and that $C(p) \leq p^{48/25+o(1)}$. No nontrivial theorems on $T(p)$ seem to be known up to this point, although Glebsky and Shparlinski ([10]) prove some relevant results when g is held fixed.

In Section 5, we investigate the number of solutions to the equations

$$(6) \quad g^h \equiv a \pmod{p^e} \quad \text{and} \quad g^a \equiv h \pmod{p^e},$$

where g is fixed and h and a are in $\{1, \dots, p^e m\}$ with much the same results as before. We also indicate how to generalize this to more equations. (Some of these results are also proved in [9].) In Section 6 we similarly investigate the equation

$$(7) \quad x^x \equiv c \pmod{p^e}$$

for fixed c , and x in $\{1, \dots, p^e(p-1)\}$, and in Section 7 we investigate the equation

$$(8) \quad h^h \equiv a^a \pmod{p^e}$$

for h and a in $\{1, \dots, p^e(p-1)\}$.

The use of the discrete exponential function $x \mapsto g^x \pmod{p}$ for g a primitive root is well known in cryptography; its inverse is commonly referred to as the discrete logarithm and computing it is one of the basic “hard problems” of public-key cryptography. (See, for example, [18, Section 3.6].) There are also uses of the function when g is not a primitive root, for example, in the Digital Signature Algorithm. (See, e.g., [18, Section 11.5].) Finally, a few cryptographic algorithms involve the self-power function $x \mapsto x^x \pmod{p}$ — notably variants of the ElGamal signature scheme, as noted in [18, Note 11.71]. The security of these cryptographic algorithms rely on the unpredictability of the inputs to these maps given the outputs. The results above and the ones in this paper go some way toward reassuring us that these maps are in fact behaving as if the inputs are randomly distributed given only basic facts known about the outputs.

2. INTERPOLATION

Let $g \in \mathbb{Z}$ be fixed and take p an odd prime. In order to count solutions to $g^x \equiv x \pmod{p^e}$, the obvious first step would be to interpolate the function $f(x) = g^x$, defined on $x \in \mathbb{Z}$, to a function on $x \in \mathbb{Z}_p$. Unfortunately, this is not possible unless $g \in 1 + p\mathbb{Z}_p$. (See for example, [11, Section 4.6], or [17, Section II.2].) However, if we “twist” the function slightly, then interpolation is possible.

To do this, let $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$ be the set of all $(p-1)$ -st roots of unity. Then for odd prime p , we have the Teichmüller character

$$\omega : \mathbb{Z}_p^\times \rightarrow \mu_{p-1},$$

which is a surjective homomorphism. It is known that \mathbb{Z}_p^\times has a canonical decomposition as $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ [11, Cor. 4.5.10], and thus for x in \mathbb{Z}_p^\times we may uniquely write $x = \omega(x) \langle x \rangle$ for some $\langle x \rangle \in 1 + p\mathbb{Z}_p$.

Proposition 2 (Prop. 4.6.3 of [11]; see also Section II.2 of [17]). *For $p \neq 2$, let $g \in \mathbb{Z}_p^\times$ and $x_0 \in \mathbb{Z}/(p-1)\mathbb{Z}$, and let*

$$I_{x_0} = \{x \in \mathbb{Z} \mid x \equiv x_0 \pmod{p-1}\} \subseteq \mathbb{Z}.$$

Then

$$f_{x_0}(x) = \omega(g)^{x_0} \langle g \rangle^x$$

defines a function on \mathbb{Z}_p such that $f_{x_0}(x) = g^x$ whenever $x \in I_{x_0}$.

In fact we can push this a little further:

Proposition 3. *Let m be any multiple of the multiplicative order of g modulo p , $p \neq 2$, such that $m \mid p-1$. Let $g \in \mathbb{Z}_p^\times$ and $x_0 \in \mathbb{Z}/m\mathbb{Z}$, and let*

$$I_{x_0} = \{x \in \mathbb{Z} \mid x \equiv x_0 \pmod{m}\} \subseteq \mathbb{Z}.$$

Then

$$f_{x_0}(x) = \omega(g)^{x_0} \langle g \rangle^x$$

defines a function on \mathbb{Z}_p such that $f_{x_0}(x) = g^x$ whenever $x \in I_{x_0}$.

Proof. Since $g^m = 1$, $\omega(g)^m = \omega(g^m) = 1$. If $x_0, x'_0 \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $x_0 \equiv x'_0 \pmod{m}$, then the two functions f_{x_0} and $f_{x'_0}$ given by Proposition 2 are equal and agree with g^x on $I_{x_0} \cup I_{x'_0}$. \square

Also, as noted for $p \neq 2$ in [11], these functions fit together into a function on $\mathbb{Z}_p \times \mathbb{Z}/m\mathbb{Z}$ defined by $F(x_1, x_0) = f_{x_0}(x_1)$, such that if $x \in \mathbb{Z}$ and $x \equiv x_0 \pmod{m}$ we have $F(x, x) = f_{x_0}(x) = g^x$. Then we have a diagram:

$$\begin{array}{ccc} \mathbb{Z}_p \times \mathbb{Z}/m\mathbb{Z} & \xrightarrow{F} & \mathbb{Z}_p^\times \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^e\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \xrightarrow{\bar{F}} & (\mathbb{Z}/p^e\mathbb{Z})^\times \end{array}$$

where the vertical arrows are the natural surjections. This commutes as a consequence of the following lemma:

Lemma 4 (Cor. 4.6.2 and just below of [11] or Lemma 2.2.5 of [16]). *For any positive integer k , $(1 + p\mathbb{Z}_p)^k \subseteq 1 + pk\mathbb{Z}_p$.*

The lemma implies that $\langle g \rangle^{p^e} \equiv 1 \pmod{p^e}$, and therefore $\langle g \rangle^x \equiv \langle g \rangle^{x'} \pmod{p^e}$ when $x \equiv x' \pmod{p^e}$. (Recall that $\mathbb{Z}_p/p^e\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}/p^e\mathbb{Z}$ for any e .)

For $p \neq 2$, if we let Δ be the diagonal inclusion map

$$\Delta : \mathbb{Z} \rightarrow \mathbb{Z}_p \times \mathbb{Z}/m\mathbb{Z}$$

given by the canonical injection $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ and the canonical surjection $\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$, then the previous diagram extends nicely to:

$$\begin{array}{ccccc}
 \mathbb{Z} & & \searrow \Delta & & \\
 \downarrow & & \mathbb{Z}_p \times \mathbb{Z}/m\mathbb{Z} & \xrightarrow{F} & \mathbb{Z}_p^\times \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{Z}/p^e\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \xrightarrow{\bar{F}} & (\mathbb{Z}/p^e\mathbb{Z})^\times \\
 & \nearrow \sim & & & \\
 \mathbb{Z}/p^em\mathbb{Z} & \xleftarrow{CRT} & & &
 \end{array}$$

where the isomorphism is given by the Chinese Remainder Theorem. Furthermore, the composition of the maps on the top line is just the map $x \mapsto g^x$ and the composition across the bottom line is the map $x \mapsto g^x \bmod p^e$:

$$\begin{array}{ccccc}
 \mathbb{Z} & & \searrow \Delta & \xrightarrow{x \mapsto g^x} & \mathbb{Z}_p^\times \\
 \downarrow & & \mathbb{Z}_p \times \mathbb{Z}/m\mathbb{Z} & \xrightarrow{F} & \downarrow \\
 & & \mathbb{Z}/p^e\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \xrightarrow{\bar{F}} & (\mathbb{Z}/p^e\mathbb{Z})^\times \\
 & \nearrow \sim & & & \downarrow \\
 \mathbb{Z}/p^em\mathbb{Z} & \xleftarrow{CRT} & & \xrightarrow{x \mapsto g^x \bmod p^e} &
 \end{array}$$

Therefore finding all solutions (x_1, x_0) to $F(x_1, x_0) \equiv x_1 \pmod{p^e}$, which is the same as finding all solutions to $f_{x_0}(x_1) \equiv x_1 \pmod{p^e}$ for all possible $x_0 \in \mathbb{Z}/m\mathbb{Z}$, will give us all solutions to $g^x \equiv x \pmod{p^e}$ as x ranges over $\mathbb{Z}/p^em\mathbb{Z}$.

3. HENSEL'S LEMMA

DEFINITION 1 (Defn. III.4.2.2 of [2]). A power series $f(x_1, x_2, \dots, x_n)$ in the ring of formal power series $\mathbb{Z}_p[[x_1, \dots, x_n]]$ with coefficients in \mathbb{Z}_p is called *restricted* if $f(x_1, \dots, x_n) = \sum_{(\alpha_i)} C_{\alpha_1, \alpha_2, \dots, \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and for every neighborhood V of 0 in \mathbb{Z}_p there is only a finite number of coefficients $C_{\alpha_1, \alpha_2, \dots, \alpha_n}$ not belonging to V (in other words, the family $(C_{\alpha_1, \alpha_2, \dots, \alpha_n})$ tends to 0 in \mathbb{Z}_p).

In particular, the series in this paper are going to be such that $C_{0,0,\dots,0} \in \mathbb{Z}_p$ and $C_{\alpha_1, \alpha_2, \dots, \alpha_n} \in p^{\alpha_1 + \alpha_2 + \dots + \alpha_n - 1} \mathbb{Z}_p$ when $\alpha_1 + \alpha_2 + \dots + \alpha_n > 0$.

In this section, we include two versions of Hensel's lemma. The first version is for n restricted power series in n unknowns.

Proposition 5 (Cor. III.4.5.2 of [2]). *Consider a collection of n restricted power series $f_j(x_1, x_2, \dots, x_n)$ for $1 \leq j \leq n$ in $\mathbb{Z}_p[[x_1, x_2, \dots, x_n]]$. Let (a_1, a_2, \dots, a_n) be*

a vector in \mathbb{Z}_p^n such that the determinant of the Jacobian matrix at (a_1, a_2, \dots, a_n)

$$\left| \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}(a_1, a_2, \dots, a_n) \right|$$

is in \mathbb{Z}_p^\times and $f_j(a_1, a_2, \dots, a_n) \equiv 0 \pmod{p}$ for $1 \leq j \leq n$. Then there exists a unique $(x_1, x_2, \dots, x_n) \in \mathbb{Z}_p^n$ for which $x_i \equiv a_i \pmod{p}$ for $1 \leq i \leq n$ and $f_j(x_1, x_2, \dots, x_n) = 0$ in \mathbb{Z}_p for $1 \leq j \leq n$.

As a corollary we get a generalization of one of the standard formulations of Hensel's Lemma to the case of restricted power series.

Corollary 6. *Let $f(x)$ be a restricted power series in $\mathbb{Z}_p[[x]]$ and a be in \mathbb{Z}_p such that $\frac{df}{dx}(a)$ is in \mathbb{Z}_p^\times and $f(a) \equiv 0 \pmod{p}$. Then there exists a unique $x \in \mathbb{Z}_p$ for which $x \equiv a \pmod{p}$ and $f(x) = 0$ in \mathbb{Z}_p .*

In our discussion of collisions below we will also need a “lifting lemma” for restricted power series of more than one variable which will allow us to count solutions modulo higher powers of p if we know the number of solutions modulo p . The following proposition, which the second author learned from Igusa's 1986 “Automorphic Forms” class at Johns Hopkins, is a generalization of the version of Hensel's Lemma in Lemma III.2.5 of [16] to the case of restricted power series, with explicit counting of the fibers.

Proposition 7. *Let $f(x_1, x_2, \dots, x_n)$ be a restricted power series in $\mathbb{Z}_p[[x_1, \dots, x_n]]$. Let*

$$N_e = \{\bar{\mathbf{a}} \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^n \mid \frac{\partial f}{\partial x_i}(\mathbf{a}) \in \mathbb{Z}_p^\times \text{ for some } 1 \leq i \leq n \text{ and } f(\mathbf{a}) \equiv 0 \pmod{p^e}\}$$

for $e > 0$, where $\bar{\mathbf{a}}$ indicates reduction of \mathbf{a} to the appropriate residue class. Then $\psi : N_{e+1} \rightarrow N_e$ is a well-defined canonical surjection with the cardinality of the fiber equal to p^{n-1} .

In particular, a point $\bar{\mathbf{a}} = (a_1, a_2, \dots, a_n) \in N_e$ can be lifted in p^{n-1} different ways to a point $\bar{\mathbf{b}} = (b_1, b_2, \dots, b_n) \in N_{e+1}$ such that $b_i \equiv a_i \pmod{p^e}$ for $1 \leq i \leq n$, so that the relationship between the cardinalities of the sets is: $|N_{e+1}| = p^{n-1}|N_e|$ for $e > 0$.

4. FIXED POINTS

Theorem 8. *For $p \neq 2$, let $g \in \mathbb{Z}_p^\times$ be fixed and let m be the multiplicative order of g modulo p . Then for every $x_0 \in \mathbb{Z}/m\mathbb{Z}$, there is exactly one solution to the equation*

$$\omega(g)^{x_0} \langle g \rangle^x = x$$

for $x \in \mathbb{Z}_p$.

Proof. We start by finding solutions modulo p . We know that $\langle g \rangle \equiv 1 \pmod{p}$, so the equation reduces to

$$\omega(g)^{x_0} \equiv x \pmod{p}.$$

For fixed g and x_0 , this obviously has exactly one solution.

Since we know that $\langle g \rangle$ is in $1 + p\mathbb{Z}_p$, we have that

$$\begin{aligned} \langle g \rangle^x &= \exp(x \log(\langle g \rangle)) = 1 + x \log(\langle g \rangle) + x^2 \log(\langle g \rangle)^2 / 2! \\ &\quad + \text{higher order terms in powers of } \log(\langle g \rangle) \end{aligned}$$

where from the definition of the p -adic logarithm we know that $\log(\langle g \rangle) \in p\mathbb{Z}_p$. Therefore we have a restricted power series and we can apply Corollary 6, which gives us a unique solution in \mathbb{Z}_p . \square

Corollary 9. *For $p \neq 2$, let $g \in \mathbb{Z}$ be fixed such that $p \nmid g$ and let m be the multiplicative order of g modulo p . Then there are exactly m solutions to the congruence*

$$(2, \text{ recalled}) \quad g^x \equiv x \pmod{p^e}$$

for $x \in \{1, 2, \dots, p^e m\}$. Furthermore, these solutions are all distinct modulo p^e and all distinct modulo m .

Proof. Theorem 8 implies that for each choice of $x_0 \in \mathbb{Z}/m\mathbb{Z}$ there is exactly one $x_1 \in \mathbb{Z}/p^e\mathbb{Z}$ with the property that

$$\omega(g)^{x_0} \langle g \rangle^{x_1} \equiv x_1 \pmod{p^e}.$$

By the Chinese Remainder Theorem, there will be exactly one $x \in \mathbb{Z}/p^e m\mathbb{Z}$ such that $x \equiv x_0 \pmod{m}$ and $x \equiv x_1 \pmod{p^e}$. By the interpolation set up since $x \equiv x_0 \pmod{m}$, we know that for this x :

$$g^x = \omega(g)^{x_0} \langle g \rangle^x \equiv x \pmod{p^e}.$$

Finally, since exactly one such x exists for each x_0 , we have our m solutions to the congruence. \square

5. TWO-CYCLES

DEFINITION 2. For a fixed prime p and for some $g \in \mathbb{Z}$, $p \nmid g$, the pair $(h, a) \in \{1, \dots, p^e(p-1)\}^2$, $p \nmid h$, $p \nmid a$ will be a *two-cycle modulo p^e* associated with g if $h \not\equiv a \pmod{p^e}$, and

$$(6, \text{ recalled}) \quad g^h \equiv a \pmod{p^e} \quad \text{and} \quad g^a \equiv h \pmod{p^e}.$$

DEFINITION 3. When we count the number of two-cycles modulo p^e , we will not distinguish between the two-cycle (h, a) and the two-cycle (a, h) . Thus, we define the number of two-cycles modulo p^e , or $|T_e|$, as

$$|T_e| = \frac{1}{2} \left| \left\{ h \in \{1, \dots, p^e(p-1)\}, p \nmid h \mid \begin{array}{l} h \not\equiv a \pmod{p^e}, \quad g^h \equiv a \pmod{p^e}, \quad \text{and} \quad g^a \equiv h \pmod{p^e} \\ \text{for some } g \in (\mathbb{Z}/p^e\mathbb{Z})^\times \text{ and } a \in \{1, \dots, p^e(p-1)\}, p \nmid a \end{array} \right\} \right|.$$

Proposition 10. *For $p \neq 2$ and a fixed $g \in \mathbb{Z}_p^\times$, let m be the multiplicative order of g modulo p . Then for every pair $(x_0, y_0) \in (\mathbb{Z}/m\mathbb{Z})^2$, there is exactly one solution to the system of equations*

$$\begin{aligned} \omega(g)^{x_0} \langle g \rangle^h &= a \\ \omega(g)^{y_0} \langle g \rangle^a &= h \end{aligned}$$

for $(h, a) \in \mathbb{Z}_p^2$.

Proof. We start by finding solutions modulo p . If we let

$$\begin{aligned} f_1(h, a) &= \omega(g)^{x_0} \langle g \rangle^h - a \\ f_2(h, a) &= \omega(g)^{y_0} \langle g \rangle^a - h \end{aligned}$$

then modulo p this system reduces to

$$\begin{aligned} f_1(h, a) &\equiv \omega(g)^{x_0} - a \pmod{p} \\ f_2(h, a) &\equiv \omega(g)^{y_0} - h \pmod{p} \end{aligned}$$

which clearly has exactly one solution $(h, a) = (\omega(g)^{x_0}, \omega(g)^{y_0})$ for fixed g , x_0 and y_0 . The power series representations for $f_1(h, a)$ and $f_2(h, a)$ are restricted power series with

$$\begin{aligned} \frac{\partial f_1}{\partial h} &= \omega(g)^{x_0} (\log(\langle g \rangle) + h \log(\langle g \rangle)^2 + \cdots) \equiv 0 \pmod{p} \\ \frac{\partial f_1}{\partial a} &= -1 \equiv -1 \pmod{p} \\ \frac{\partial f_2}{\partial h} &= -1 \equiv -1 \pmod{p} \\ \frac{\partial f_2}{\partial a} &= \omega(g)^{y_0} (\log(\langle g \rangle) + a \log(\langle g \rangle)^2 + \cdots) \equiv 0 \pmod{p}. \end{aligned}$$

Thus the determinant of the Jacobian matrix is congruent to -1 modulo p and by Proposition 5 the unique solution modulo p to this system lifts to a unique solution $(h, a) \in \mathbb{Z}_p^2$. \square

Proposition 11. *For $p \neq 2$ and a fixed $g \in \mathbb{Z}$, $p \nmid g$, let m be the multiplicative order of g modulo p . Then if*

$$\begin{aligned} |T_{e,g}| &= \frac{1}{2} \left| \left\{ h \in \{1, \dots, p^e m\}, p \nmid h \mid \begin{array}{l} h \not\equiv a \pmod{p^e}, \\ g^h \equiv a \pmod{p^e}, \text{ and } g^a \equiv h \pmod{p^e} \end{array} \right\} \right| \\ &\quad \text{for some } a \in \{1, \dots, p^e m\}, p \nmid a \Big|. \end{aligned}$$

is the number of two-cycles modulo p^e associated with that particular g ,

$$|T_{e,g}| = (m^2 - m)/2.$$

Proof. Parallel to the proof of Corollary 9, for each choice of (x_0, y_0) in $(\mathbb{Z}/m\mathbb{Z})^2$, Proposition 10 gives us exactly one pair (h, a) in $(\mathbb{Z}/p^e m\mathbb{Z})^2$ satisfying $g^h \equiv a \pmod{p^e}$ and $g^a \equiv h \pmod{p^e}$. Thus there are m^2 such pairs total, but m of them correspond to the case where $h \equiv a \pmod{p^e}$. Dividing by 2 to account for swapping the roles of h and a gives us the proposition. \square

Theorem 12. *For a given prime $p \neq 2$, the number of two-cycles $|T_e|$ is*

$$|T_e| = \sum_{m|(p-1)} \phi(m) p^{e-1} (p-1)(m-1)/2.$$

Proof. First note that if an h in $\{1, \dots, p^e m\}$ forms part of a two-cycle associated with g and a , then the values in $\{1, \dots, p^e(p-1)\}$ which do the same will be exactly those which are congruent to h modulo p^e and modulo m , and thus modulo $p^e m$. So each element of $T_{e,g}$ gives rise to exactly $(p-1)/m$ elements of T_e in this fashion. On the other hand, if some a in $\{1, \dots, p^e(p-1)\}$ forms part of a two-cycle associated with h and g , then so will an a in $\{1, \dots, p^e m\}$ which is congruent to it modulo

$p^e m$. So each element of $T_{e,g}$ gives rise to only one element of T_e in this fashion. Therefore we have

$$\begin{aligned} |T_e| &= \sum_{g \in (\mathbb{Z}/p^e\mathbb{Z})^\times} \left(\frac{p-1}{m} \right) |T_{e,g}| \\ &= \sum_{m|(p-1)} \phi(m) p^{e-1} (p-1)(m-1)/2. \end{aligned}$$

□

Alternatively, we can count rooted closed walks rather than cycles, a viewpoint which in some ways lends itself better to generalizations.

DEFINITION 4. For a fixed prime p and for some $g \in \mathbb{Z}$, $p \nmid g$, the ordered tuple (h_1, \dots, h_k) is a *rooted closed walk of length k modulo p^e associated with g* if the k equations

$$\begin{aligned} g^{h_1} &\equiv h_2 \pmod{p^e}, \\ g^{h_2} &\equiv h_3 \pmod{p^e}, \\ &\vdots \\ g^{h_{k-1}} &\equiv h_k \pmod{p^e}, \\ g^{h_k} &\equiv h_1 \pmod{p^e} \end{aligned}$$

are satisfied.

Then Corollary 9 is equivalent to saying that there are exactly m rooted closed walks of length 1 associated with g in $\{1, 2, \dots, p^e m\}$, and Proposition 11 is equivalent to saying that there are m^2 rooted closed walks of length 2 associated with g (including the fixed points) in $\{1, 2, \dots, p^e m\}^2$. In an exactly parallel manner, we can prove the following generalization:

Theorem 13. *For $p \neq 2$ and a fixed $g \in \mathbb{Z}$, $p \nmid g$, let m be the multiplicative order of g modulo p . Then there are exactly m^k rooted closed walks of length k modulo p^e associated with g in $\{1, 2, \dots, p^e m\}^k$. Furthermore, any two of these rooted closed walks are distinct modulo p^e and distinct modulo m .*

REMARK 1. In the case where $m = p - 1$, this is an equivalent statement to Theorem 1 of [9], where it is proved using purely combinatorial methods. For general m , our statement implies that of [9].

6. SELF-POWER SOLUTIONS

We now turn to the function $x \mapsto x^x \pmod{p}$, which is sometimes known as the *self-power map*.

The proof of the following elementary lemma was essentially worked out in Theorem 2 of [8].

Lemma 14. *For $p \neq 2$, let $c \in (\mathbb{Z}/p\mathbb{Z})^\times$ be fixed and let m be the multiplicative order of c modulo p . Also fix $x_0 \in \{0, 1, \dots, p-2\}$. Then the number of solutions $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ to the equivalence*

$$x^{x_0} \equiv c \pmod{p}$$

is

$$\begin{cases} \gcd(x_0, p-1) & \text{if } \gcd(x_0, p-1) \mid \frac{p-1}{m}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a fixed integer t the set of t -th powers, $P_t = \{x^t \mid x \in (\mathbb{Z}/p\mathbb{Z})^\times\}$ forms a subgroup of index $\gcd(t, p-1)$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. Using our set cardinality notation, we have that $|P_t| = (p-1)/\gcd(t, p-1)$. If $\gcd(x_0, p-1) \nmid \frac{p-1}{m}$ then c is not in P_{x_0} , so $x^{x_0} \equiv c \pmod{p}$ cannot have any solutions. Otherwise, any element of P_{x_0} is an x_0 -th power in exactly $\gcd(x_0, p-1)$ ways, so the equivalence has exactly $\gcd(x_0, p-1)$ solutions. \square

Corollary 15. *For $p \neq 2$, let $c \in (\mathbb{Z}/p\mathbb{Z})^\times$ be fixed and let m be the multiplicative order of c modulo p . Then the number of solutions $x \in \{1, 2, \dots, p(p-1)\}$ to the equivalence $x^x \equiv c \pmod{p}$ such that $p \nmid x$ is given by the formula:*

$$\sum_{\substack{0 \leq x_0 \leq p-2 \\ \gcd(x_0, p-1) \mid \frac{p-1}{m}}} \gcd(x_0, p-1) = \sum_{d \mid \frac{p-1}{m}} d \phi\left(\frac{p-1}{d}\right).$$

Proposition 16. *For $p \neq 2$, let $c \in \mathbb{Z}_p^\times$ be fixed and let m be the multiplicative order of c modulo p . Then for fixed $x_0 \in \mathbb{Z}/(p-1)\mathbb{Z}$, the number of solutions to the equation*

$$\omega(x)^{x_0} \langle x \rangle^x = c$$

for $x \in \mathbb{Z}_p^\times$ is

$$\begin{cases} \gcd(x_0, p-1) & \text{if } \gcd(x_0, p-1) \mid \frac{p-1}{m}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a fixed x_0 , we consider the function

$$f(x) = \omega(x)^{x_0} \langle x \rangle^x - c$$

and look for solutions $x \in \mathbb{Z}_p^\times$ to $f(x) \equiv 0 \pmod{p}$. Since we know that $\langle x \rangle$ is in $1 + p\mathbb{Z}_p$, we have that

$$\begin{aligned} \langle x \rangle^x = \exp(x \log(\langle x \rangle)) &= 1 + x \log(\langle x \rangle) + x^2 \log(\langle x \rangle)^2 / 2! \\ &+ \text{higher order terms in powers of } x \log(\langle x \rangle) \end{aligned}$$

where from the definition of the p -adic logarithm we know that $\log(\langle x \rangle) \in p\mathbb{Z}_p$. Now if we consider the power series representation of $f(x)$, we see that

$$\begin{aligned} f(x) = \omega(x)^{x_0} - c &+ \omega(x)^{x_0} x \log(\langle x \rangle) \\ &+ \text{higher order terms in } p^2\mathbb{Z}_p. \end{aligned}$$

Since ω is constant on each of the $p-1$ disjoint cosets of $p\mathbb{Z}_p$ that cover \mathbb{Z}_p^\times or see [17, Prop.2, Section IV.2], we have that

$$\frac{df}{dx} = \omega(x)^{x_0} [\log(\langle x \rangle) + 1] \equiv \omega(x)^{x_0} \pmod{p}$$

since $\log(\langle x \rangle) \in p\mathbb{Z}_p$. As $\omega(x)^{x_0} \not\equiv 0 \pmod{p}$, we have by Corollary 6 that the number of solutions in \mathbb{Z}_p is the same as the number of solutions in Lemma 14. \square

Corollary 17. For $p \neq 2$, let $c \in \mathbb{Z}_p^\times$ be fixed and let m be the multiplicative order of c modulo p . Then the number of solutions to the congruence

$$(7, \text{recalled}) \quad x^x \equiv c \pmod{p^e}$$

for x such that $x \in \{1, 2, \dots, p^e(p-1)\}$, $p \nmid x$, is given by the formula:

$$\sum_{\substack{0 \leq x_0 \leq p-2 \\ \gcd(x_0, p-1) \mid \frac{p-1}{m}}} \gcd(x_0, p-1) = \sum_{d \mid \frac{p-1}{m}} d \phi\left(\frac{p-1}{d}\right).$$

Proof. The proof is parallel to that of Corollary 9. \square

7. COLLISIONS

DEFINITION 5. The set of solutions (h, a) , where h and $a \in \{1, 2, \dots, p(p-1)\}$, $p \nmid h$ and $p \nmid a$, to the equivalence

$$h^h \equiv a^a \pmod{p}$$

will be denoted C_1 for *collisions* and we will use the notation $|C_1|$ for the number of such collisions. More generally, we will use the notation $|C_e|$ to denote the number of *collisions* (h, a) , where h and $a \in \{1, 2, \dots, p^e(p-1)\}$, $p \nmid h$ and $p \nmid a$, which are solutions to the equivalence

$$h^h \equiv a^a \pmod{p^e}.$$

Recall that \bar{x} indicates reduction of x to the appropriate residue class.

Lemma 18. For fixed x_0 and $y_0 \in \{0, 1, \dots, p-2\}$, if

$$N_1^\times = \{(x, y) \in ((\mathbb{Z}/p\mathbb{Z})^\times)^2 \mid x^{x_0} - y^{y_0} = 0 \text{ in } \mathbb{Z}/p\mathbb{Z}\},$$

then

$$|N_1^\times| = (p-1) \gcd(x_0, y_0, p-1).$$

Proof. For a fixed integer t the set of t -th powers, $P_t = \{x^t \mid x \in (\mathbb{Z}/p\mathbb{Z})^\times\}$ forms a subgroup of index $\gcd(t, p-1)$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. Using our set cardinality notation, we have that $|P_t| = (p-1)/\gcd(t, p-1)$. Let $\mathfrak{J} = P_{x_0} \cap P_{y_0}$, then \mathfrak{J} is a subgroup of order

$$|\mathfrak{J}| = \gcd(|P_{x_0}|, |P_{y_0}|) = \frac{(p-1) \gcd(x_0, y_0, p-1)}{\gcd(x_0, p-1) \gcd(y_0, p-1)}.$$

Now, we need to count all $(x, y) \in ((\mathbb{Z}/p\mathbb{Z})^\times)^2$ such that $x^{x_0} \equiv y^{y_0} \pmod{p}$. If $x^{x_0} \equiv y^{y_0} \pmod{p}$ then x^{x_0} and y^{y_0} are in the set \mathfrak{J} above. Thus, we have that

$$|N_1^\times| = \sum_{i \in \mathfrak{J}} |\{x \in (\mathbb{Z}/p\mathbb{Z})^\times \mid x^{x_0} \equiv i \pmod{p}\}| \cdot |\{y \in (\mathbb{Z}/p\mathbb{Z})^\times \mid y^{y_0} \equiv i \pmod{p}\}|.$$

For each $i \in \mathfrak{J}$, $|\{x \in (\mathbb{Z}/p\mathbb{Z})^\times \mid x^{x_0} \equiv i \pmod{p}\}| = \gcd(x_0, p-1)$. So that

$$|N_1^\times| = |\mathfrak{J}| \cdot \gcd(x_0, p-1) \cdot \gcd(y_0, p-1) = (p-1) \gcd(x_0, y_0, p-1).$$

\square

Proposition 19. For $p \neq 2$ and for fixed x_0 and $y_0 \in \mathbb{Z}/(p-1)\mathbb{Z}$, if we consider the function $f(h, a) = \omega(h)^{x_0} \langle h \rangle^h - \omega(a)^{y_0} \langle a \rangle^a$ for $h, a \in \mathbb{Z}_p^\times$ and let

$$|N_1^\times| = |\{(\bar{h}, \bar{a}) \in ((\mathbb{Z}_p/p\mathbb{Z}_p)^\times)^2 \mid f(h, a) \equiv 0 \pmod{p}\}|,$$

then

$$|N_1^\times| = (p-1) \gcd(x_0, y_0, p-1)$$

Proof. For a fixed x_0 and y_0 , we look for solutions $h, a \in \mathbb{Z}_p^\times$ to $f(h, a) \equiv 0 \pmod{p}$. Since we know that $\langle h \rangle$ and $\langle a \rangle$ are elements in $1 + p\mathbb{Z}_p$, we have that

$$\begin{aligned} \langle h \rangle^h = \exp(h \log(\langle h \rangle)) &= 1 + h \log(\langle h \rangle) + h^2 \log(\langle h \rangle)^2 / 2! \\ &+ \text{higher order terms in powers of } h \log(\langle h \rangle) \end{aligned}$$

where from the definition of the p -adic logarithm we know that $\log(\langle h \rangle) \in p\mathbb{Z}_p$. Now if we consider the number of solutions $|N_1^\times|$ using the power series representation of $f(h, a)$, we see that

$$(9) \quad f(h, a) = \omega(h)^{x_0} - \omega(a)^{y_0} + \text{higher order terms in } p\mathbb{Z}_p.$$

In this way, we see that

$$|N_1^\times| = |\{(\bar{h}, \bar{a}) \in ((\mathbb{Z}/p\mathbb{Z})^\times)^2 \mid \omega(h)^{x_0} - \omega(a)^{y_0} \equiv 0 \pmod{p}\}|.$$

From this expression and Lemma 18, we have that

$$|N_1^\times| = (p-1) \gcd(x_0, y_0, p-1).$$

□

Corollary 20. For $p \neq 2$, the number of collisions (h, a) for h and $a \in \{1, 2, \dots, p(p-1)\}$ such that $p \nmid h$, $p \nmid a$, and $h^h \equiv a^a \pmod{p}$ is given by the formula:

$$|C_1| = \sum_{0 \leq x_0, y_0 \leq p-2} (p-1) \gcd(x_0, y_0, p-1) = (p-1) \sum_{d \mid p-1} d J_2((p-1)/d)$$

where $J_2(n) = n^2 \prod_{p \mid n} (1 - p^{-2})$ is Jordan's totient function, which counts the number of pairs of positive integers all less than or equal to n that form a mutually coprime triple together with n .

Proposition 21. For $p \neq 2$ and for fixed x_0 and $y_0 \in \mathbb{Z}/(p-1)\mathbb{Z}$, if we consider the function $f(h, a) = \omega(h)^{x_0} \langle h \rangle^h - \omega(a)^{y_0} \langle a \rangle^a$ for $h, a \in \mathbb{Z}_p^\times$ and let

$$N_e^\times = \{(\bar{h}, \bar{a}) \in ((\mathbb{Z}_p/p^e\mathbb{Z}_p)^\times)^2 \mid f(h, a) \equiv 0 \pmod{p^e}\},$$

then

$$|N_e^\times| = p^{e-1} |N_1^\times|.$$

Proof. Looking more carefully at our series representation for $f(h, a)$ in Equation 9 from Proposition 19, we have that

$$\begin{aligned} f(h, a) &= \omega(h)^{x_0} - \omega(a)^{y_0} + \omega(h)^{x_0} h \log(\langle h \rangle) - \omega(a)^{y_0} a \log(\langle a \rangle) \\ &+ \text{higher order terms in } p^2\mathbb{Z}_p. \end{aligned}$$

Since ω is constant on each of the $p-1$ disjoint cosets of $p\mathbb{Z}_p$ that cover \mathbb{Z}_p^\times or see [17, Prop.2, Section IV.2], we have that

$$\frac{\partial f}{\partial h} = \omega(h)^{x_0} [\log(\langle h \rangle) + 1] \equiv \omega(h)^{x_0} \pmod{p}$$

since $\log(\langle h \rangle) \in p\mathbb{Z}_p$. As $\omega(h)^{x_0} \not\equiv 0 \pmod{p}$, we have by Proposition 7 with $n = 2$ that

$$|N_e^\times| = p|N_{e-1}^\times|.$$

for $e > 1$, and our Proposition follows. \square

Corollary 22. *For $p \neq 2$, there are exactly $|C_e| = p^{e-1}|C_1|$ collisions that are solutions to the congruence*

$$(8, \text{recalled}) \quad h^h \equiv a^a \pmod{p^e}$$

for (h, a) such that h and $a \in \{1, 2, \dots, p^e(p-1)\}$, $p \nmid h$, $p \nmid a$.

Proof. The proof is parallel to that of Corollary 9. \square

REMARK 2. Note that Corollaries 20 and 22 could also have been proved by squaring the results of Corollaries 15 and 17, respectively, and summing over all c .

8. CONCLUSIONS AND FUTURE WORK

Previous work on solutions to (1) and related equations has focused on finding how primitive roots modulo p , or specified powers of primitive roots, are distributed in arithmetic progressions contained in $\{1, \dots, p\}$ with differences dividing $p-1$. We hope that this paper shows that another course might also be fruitful: start with the solutions to an exponential equation which are in $\{1, \dots, p(p-1)\}$ (or $\{1, \dots, p^e(p-1)\}$) and determine how they are distributed among the subintervals of length p (or p^e). Furthermore, we think the use of p -adic numbers also suggests new lines of attack that may be useful in the future. For example, the ability to extend the p -adic exponential function to rings of integers in extension fields of \mathbb{Q}_p might provide a useful way of looking at, or even posing, new problems in finite field extensions of $\mathbb{Z}/p\mathbb{Z}$.

In future extensions of this work we hope to consider solutions of more exponential equations, including the equation

$$(10) \quad h^{h/d} \equiv a^{a/d} \pmod{p^e}, \quad d = \gcd(h, a, p-1)$$

considered (with $e = 1$) in [15] as closely related to (3). Another problem that should be tractable using our methods is finding solutions of

$$(11) \quad g^{x-1+c} \equiv x \pmod{p^e}$$

for c fixed. This was raised in [7] (with $e = 1$) as related to ‘‘Golumb rulers’’, which have applications in error correction and in controlling the effects of electromagnetic signals interference. Finally, one could consider the ‘‘discrete Lambert’’ map $x \mapsto xg^x$ for g fixed, which is related to the standard ElGamal signature scheme and the Digital Signature Algorithm in a similar fashion to the way the self-power function is related to its variants. Then one could ask for solutions of

$$(12) \quad xg^x \equiv c \pmod{p^e}$$

for fixed c , or collisions of the discrete Lambert map, namely solutions of

$$(13) \quad hg^h \equiv ag^a \pmod{p^e}.$$

Finally, for completeness one should investigate the situation when $p = 2$. Counting solutions modulo p is trivial in this case, but the p -adic situation is slightly more complicated than in the $p \neq 2$ case.

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